

Demand-driven Optimal Seat Allocation

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Abstract: We study the air ticket inventory competition of two flights and two levels of air ticket price for each flight. Each flight has a fixed initial capacity and prices and competes in setting the optimal stopping region to maximize the total expected revenue over a finite sales horizon. Customers are divided into three types according to their desire. The choice behavior can be described as a horizontal diversion and vertical diversion of demand; the choice probability depends on the current prices. Assuming the arrival process is homogeneous Poisson distribution, we presented the seat allocation model and found that Nash equilibrium was applicable in the allocation model. The optimal policy derived is a threshold policy, which simplified the model and improved computational efficiency. Numerical experiments describe the application of the model in the real world and the strong linkage between the threshold, rival's strategy and ratio of different passengers.

1. Introduction

Revenue management (RM) is an effective means of increasing revenue for airlines. Feldman(1991)^[1] has pointed out that the use of revenue management system will increase by 2 % – 7 % of the revenues. It can help airlines set the right price and right number seats at the right time for the right customers to maximize revenue. Nevertheless, traditional RM models usually assume an industry monopoly, which does not work effectively in reality. See the airline flight schedules in Table 1, CA, CZ and MU schedule flights between Shanghai and Beijing, nearly the same times, aircraft and even the price for advance-purchase tickets. Direct competition encourages airlines to review their RM policy: during decision-making, they should take not only their own demand and capacity into account but also the response from their rivals, especially the influence of their rivals on seat allocation.

Table 1. Flight Schedule for Airlines from Shanghai to Beijing, March 20, 2018

Airline	Flight	Aircraft	Departure	Price
CA	1590	Airbus 330	8:55	810
MU	5103	Airbus 330	9:00	827
CZ	5140	Airbus 330	10:00	810
MU	5105	Airbus 330	10:00	827
CA	1832	Airbus 330	10:55	810
MU	5107	Airbus 330	11:00	827

Demand becomes more uncertain under competition. Passengers will overflow to other flights if their initial demand is unavailable; initial high fare demand will buy-down when lower class is available and initial low fare demand will buy-up when low-fare class closed. We define the former diversion of demand between flights' horizontal diversion and the latter among classes' vertical diversion. Besides, there still exists an oblique diversion between classes belonging to different flights.

In this paper, we study a single-leg, two-flight and two-fare class seat allocation problem between a single origin–destination pair. Booking requests arrive according to a random process and each customer requests only one seat. The competition and demand diversion are shown in Figure 1. In the aviation market, airlines can acquire real-time information of the seat inventory and price with an acceptable cost, so we study competing airlines engaging in a non-cooperative game with complete information.

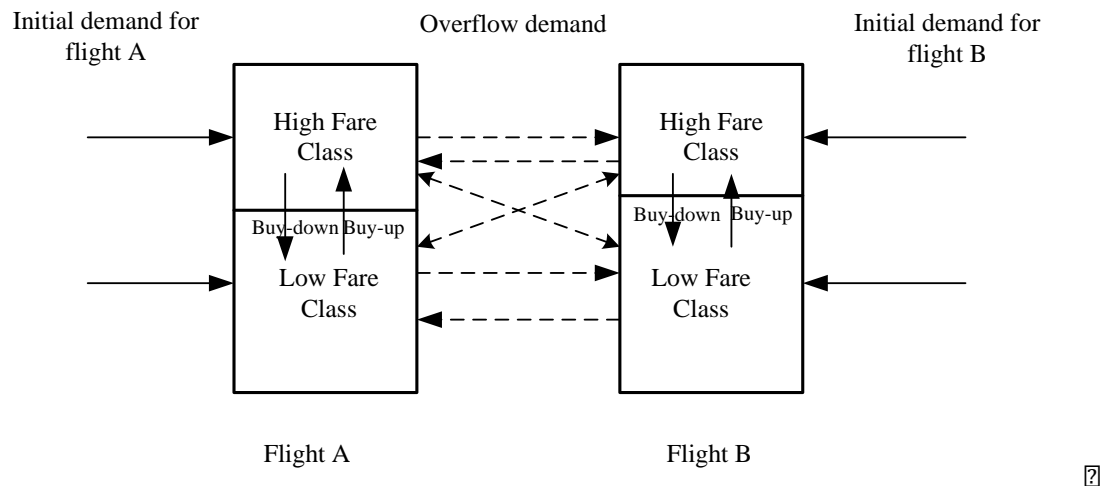


Figure 1. Demand Diversion between Two Flights and Two Classes under Competition

Seat allocation is the crucial part of revenue management. Most of the research assumes that the market is monopolistic. These monopoly airlines use inventory control as a tool to allocate seats to different levels of demand optimally to maximize the total expected revenue. Literature on seat allocation under competition is not so sufficient, and all the literature assumes that price is fixed, which is similar to general inventory competition model. Parlar (1988) employs game theory to analyze inventory management with demand substitution. He assumes that each retailer knows the substitution rate and demand density. Zhao and Atkins (2000) develop the model of inventory for a single class of two airlines under competition. Netessin and Shumsky (2005) examine the seat inventory control problem under both horizontal competition (two airlines compete for passengers on the same flight leg) and vertical competition (different airlines fly different legs on a multi-leg itinerary). They provide a general sufficient condition under which a pure-strategy Nash equilibrium exists in these revenue management games. The analytical results demonstrate that more seats are protected for higher-fare passengers under horizontal competition comparing to monopoly single airline acts as a monopoly. However, under horizontal competition, the result is totally different. The literature mentioned above focuses on inventory control policy for two flights, served single-leg. Jiang and Pang (2011) extend the model from single-leg to airline network. Our model settled in this paper is dynamic, and decision will be optimal to real-time.

Recently, revenue management competition in both inventory and pricing has been paid more and more attention. In this paper, we aim at modeling the seat allocation problem with passenger horizontal and vertical diversions (it is not same to vertical competition in Netessin and Shumsky [2005]) under competition. Demands for the two fare classes arrive concurrently, which is closer to the truth. The issue is the correct timing of closing the low-fare class given his rival's stopping region.

Our model assumes that the decision rule is dynamic: the optimal region to close the low-fare class depends on the number of seats left and the rival's strategy. An airline can adjust or decide whether to close the low fare in real time to maximize the total expected revenue (TER). Like static model, our optimal policy is threshold policy, which simplified the model and improved computational efficiency. The no-reopen assumption of low-fare class made our model more practical. Unlike the static model, the optimal policy is not fixed in advance; it is made dynamically according to the current situations and rival's strategy to mitigate risk. Our contribution has established a model involving the horizontal and vertical diversion synchronously.

2. Program Description and Assumptions

Suppose two airlines are offering direct flights between the same origin and destination, with departures and arrivals at similar times. We assume that other flights on this route are scheduled sufficiently far away in time so that they can be ignored. For simplicity, suppose that both flights have the same capacity and there are only two fare classes available for passengers: low-fare class and high-fare class. Customers are segmented into low-fare and high fare fare, there will be a low fare demand. A customer's reservation price is specific to every individual demands by their reservation prices: if his reservation price equals or exceeds the low fare and is less than the high and is confidential to the airlines. However, the air tickets'seller can distigush the distribution of the reservation prices from the passangers' preference.

Supposing that two airlines with the same capacity C , two fare classes charged with prices r_1 and r_2 ($r_1 > r_2$), denoting the low fare price and the high fare price, respectively. Each customer has a reservation price v , and v is independent identical distribution (i.i.d), the cumulative distribution function is denoted by $F(v)$, which is continuous and differentiable. To be simple we assume that $F(v)$ is strictly increasing. The purchasing probability given price r_i is $\Pr(v \geq r_i) = 1 - F(r_i)$. To simplify, Figure 2 is used to substitute for Figure 1. There are three types of passengers: type 1 and type 2 are respectively for flight A and flight B only; whereas type 3 is flexible and willing to take either of the two flights. Passengers of type 3 are flexible because they are indifferent between two flights as long as they can book a seat at the lower fare class they want. If fares of the two flights are equal, the proportions of three types of passenger flow are denoted by α_i ($0 < \alpha_i < 1$), and $\sum \alpha_i = 1$. For convenience, we assume that the arrival process of all three types of passengers are homogeneous Poisson distribution and independent of each other. Let λ be the total arrival rate of the passengers, and $\lambda_i = \alpha_i * \lambda$ be the arrival rate of type i passengers. At the beginning of the sales horizon, both low-fare and high-fare classes are open to book. P_A and P_B are the lowest prices of currently available classes of flights A and B. When there comes a request of type 1 (or type 2), he will purchase flight A (or B) when his reservation price $v \geq P_A$ (or $v \geq P_B$). When low-fare class is still available ($P_A = P_B$), the original demands for high-fare class will buy down with the probability p_{down} , and if the low-fare class is closed (we assumed that low-fare class will not reopen once closed in later periods), then the original demands for low-fare class will buy up for high-fare class with the probability p_{up} . If a passenger is type 3, he will purchase the lower fare flight by the rules as follows:

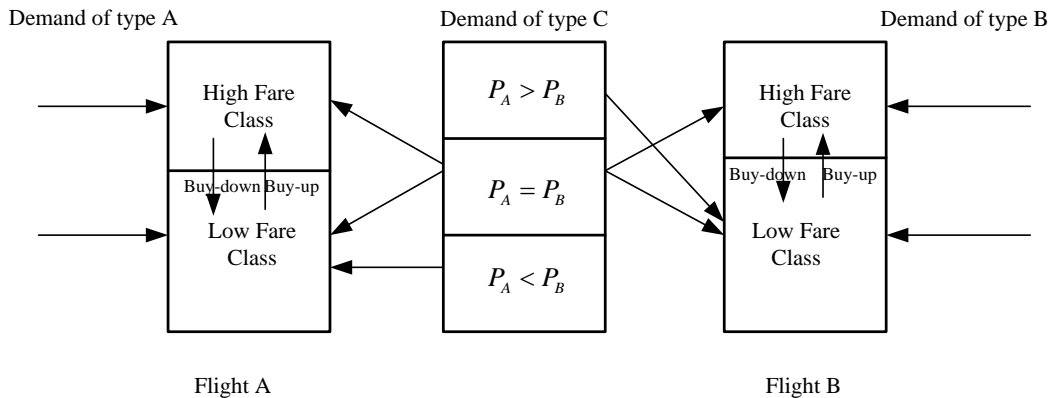


Figure 2. Demand Diversions of Three Types of Passengers

(1) $P_A \neq P_B$, and if his reservation price $v \geq (P_A \wedge P_B)$, $(P_A \wedge P_B) = \min(P_A, P_B)$, he will buy the lower fare flight;

(2) $P_A = P_B$, and if his reservation price $v \geq P_A$ (and P_B), he will book flight A with probability $\lambda_3 * \Pr\{v \geq (P_A \wedge P_B)\}$, and flight B with probability $1 - \beta$.

3. The Basic Model with Horizontal Diversion Only

In this section we analyze the basic model, which assumes that there is no vertical diversion of demand $p_{down} = p_{up} = 0$

3.1 Formulation

We Let N be the total number of sale periods, and periods are indexed consecutively by periods remaining, i.e. period N is the starting period and period 0 is the ending period. At any period $n \in [N, 0]$, let $y(n)$ be the number of seats left at n . Our control policy is whether to close the low-fare class. Obviously, this should be based on $y(n)$, the number of seats left, as well as n , the time left before departure. Thus, it is an optimal stopping problem. A stopping rule can be characterized by a stopping region $y^*(n)$ contained in $\{0, 1, 2, \dots, C\} * [N, 0]$.

The probability of type 1 customers arrival and choose the low-fare class of flight A is $\lambda_1 * \Pr\{r_2 \leq v \leq r_1\}$, and the probability of high-fare class is $\lambda_1 * \Pr\{v \geq r_1\}$. So as type 2 for flight B, the probability of choosing low-fare class is $\lambda_2 * \Pr\{r_2 \leq v \leq r_1\}$ and choosing high-fare class is $\lambda_2 * \Pr\{v \geq r_1\}$. Customers of type 3 are flexible and price sensitive; they will book the low-fare class if it is available whether their reservation prices are low or high. So the probability of type 3 choosing the lower fare flight A or B is $\lambda_3 * \Pr\{v \geq (P_A \wedge P_B)\}$.

Considering the arrival process and choice behavior of potential customers, we can get the effective demands for low-fare class and high-fare class of flight A and flight B, respectively, as follows:

$$P_{AL}^n = \lambda_1 * [F(r_2) - F(r_1)] + \lambda_3 * F(r_2) * I_{(P_A < P_B)} + \beta * \lambda_3 * F(r_2) * I_{(P_A = P_B)} \quad (1)$$

$$P_{AH}^n = \lambda_1 * F(r_1) \quad (2)$$

$$P_{BL}^n = \lambda_2 * [F(r_2) - F(r_1)] + \lambda_3 * F(r_2) * I_{(P_A > P_B)} + (1 - \beta) * \lambda_3 * F(r_2) * I_{(P_A = P_B)} \quad (3)$$

$$P_{BH}^n = \lambda_2 * F(r_1) \quad (4)$$

where $I_{\{condition\}}$ is an indicator function, and $I_{\{condition\}} = \begin{cases} 1 & \text{if condition is true} \\ 0 & \text{otherwise} \end{cases}$

Clearly, the difference of seat inventory control between monopoly and competitive markets is that demand for flight A or B is affected by not only its own seat availability but also by the other's.

Let $F_A(n, y_A)$, $F_B(n, y_B)$ be the maximal expected revenue from period n to 0 when the remaining seats on flight A and B is y ; here we assume that low-fare class is still open, then the dynamic programming equation of this problem is presented as follows:

$$F_A(n, y_A) = P_{AL}^n * [r_2 + F_A(n-1, y_A-1)] + P_{AH}^n * [r_1 + F_A(n-1, y_A-1)] + (1 - P_{AL}^n - P_{AH}^n) * F_A(n-1, y_A) \quad (5)$$

$$F_B(n, y_B) = P_{BL}^n * [r_2 + F_B(n-1, y_B-1)] + P_{BH}^n * [r_1 + F_B(n-1, y_B-1)] + (1 - P_{BL}^n - P_{BH}^n) * F_B(n-1, y_B) \quad (6)$$

Here the first term is the expected total revenue if a low-fare ticket is sold at period n , and the

second is the expected total revenue if a high-fare ticket is sold, and the third is the expected total revenue otherwise.

Let $G_A(n, y_A)$ $G_B(n, y_B)$ be the expected revenue over n to 0 with y seats left at n and low-fare class closed already by period n , so

$$G_A(n, y_A) = (\lambda_1 + \beta * \lambda_3 * I_{(P_A=P_B)}) * \Pr(r_1) * [r_1 + G_A(n-1, y_A-1)] + [1 - (\lambda_1 + \beta * \lambda_3 * I_{(P_A=P_B)}) * \Pr(r_1)] * G(n-1, y_A) \quad (7)$$

$$G_B(n, y_B) = [\lambda_1 + (1-\beta) * \lambda_3 * I_{(P_A=P_B)}] * \Pr(r_1) * [r_1 + G_B(n-1, y_B-1)] + [1 - (\lambda_1 + (1-\beta) * \lambda_3 * I_{(P_A=P_B)}) * \Pr(r_1)] * G(n-1, y_B) \quad (8)$$

The first item of the right-hand side means revenue gains when a ticket is sold at a high fare in period n , and the second item means revenue gains otherwise.

Then the optimal stopping region $y^*(n)$ can be characterized by

$$y_A^*(n) = \max \{y_A \mid G_A(n, y_A) \geq F_A(n, y_A)\} \quad (10)$$

$$y_B^*(n) = \max \{y_B \mid G_B(n, y_B) \geq F_B(n, y_B)\} \quad (11)$$

On the other hand, if $y_A(n) \neq y_A^*(n)$, $y_B(n) \neq y_B^*(n)$, $F_A(n, y_A) > G_A(n, y_A)$, $F_B(n, y_B) > G_B(n, y_B)$, it is optimal to keep the low-fare class open.

3.2 Structure of optimal seat allocation policies

Airlines decide whether to close the low-fare class at the beginning of each period n to maximize their total expected revenue (TER). They decide according to the current market states (seats left at period n (y_A, y_B) on flights A and B) and forecast of demands arrival. They will increase the yield of a ticket and reduce one seat if passengers buy one; otherwise they will keep the seats for the next period.

At any period n , ($0 < n < N$), airline will sell out 1 ticket at most, and the rest will be kept for the next period. Airline A (or B) has to decide whether to close the low-fare class or not. We divide situations into three kinds as follows based on seats left on two flights (y_A, y_B).

Case 1 $y_A = y_B = 0$, If $y_B = 0$ (or $y_A = 0$), seats are not available of flight B (or A), so let $P_B = \infty$ (or $P_A = \infty$). There was no increase in revenue, so both airlines should close the sale horizon, and $F_A(n, 0) = F_B(n, 0) = 0$.

Case 2 $y_A > 0, y_B = 0$ (or $y_A = 0, y_B > 0$), there are still seats left on flight A (or B) but no seats on flight B (or A), airline A (or B) decides to stop the low fare or not. All flexible passengers of type 3 with reservation prices that exceed the given price will turn to flight A (or B), so the potential demand rate is $(\lambda_1 + \lambda_3)$ or $(\lambda_2 + \lambda_3)$.

We take airline A for example; it is similar to airline B. When low-fare class is still open on flight A, the TER expression is as follows, (12)

$$F_A(n, y_A) = \{\lambda_1 * [\Pr(r_2) - \Pr(r_1)] + \lambda_3 * \Pr(r_2)\} * [r_2 + F_A(n-1, y_A-1)] + \lambda_1 * F(r_1) * [r_1 + F_A(n-1, y_A-1)] + [1 - (\lambda_1 + \lambda_3) * \Pr(r_2)] * F_A(n-1, y_A)$$

If low-fare class has closed and only high-fare class is open, the revenue will be

$$G_A(n, y_A) = (\lambda_1 + \lambda_3) * F(r_1) * [r_1 + G_A(n-1, y_A-1)] + (1 - (\lambda_1 + \lambda_3) * F(r_1)) * G(n-1, y_A) \quad (13)$$

The optimal stopping region is $y_A^*(n) = \max\{y_A | G_A(n, y_A) \geq F_A(n, y_A)\}$ as monopoly market, otherwise keep low-fare class open.

Case 3 $y_A > 0, y_B > 0$, there are still seats left on both airlines A and B, the optimal problems faced by airlines are when to close the low-fare class (we assume that low-fare class is still open at the beginning of period n). Figure 3 shows that if airline B decides to stop low-fare class earlier and keeps more seats for high $y_B^*(n) > y_A^*(n)$, flexible passengers will turn to airline A before A's stopping region because P_A is lower until $P_A = P_B$, and vice-versa.

We assume that both low-fare and high-fare classes are open at the beginning of period n on both flights A and B. Airlines can sell a ticket as low or high-fare class according to the demand arriving at period n .

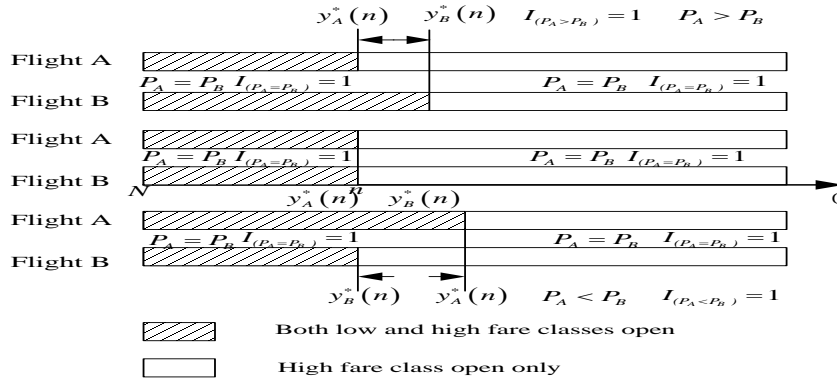


Figure 3. Decision-making of Stopping Region

The TER that airline A can obtain with two fare classes open can be calculated by (5), and high-fare class open only by equation (6)

We can get the optimal stopping region of airline A given airline B's stopping region firstly by (10); similarly, the optimal stopping region of airline B can be calculated by equation (11).

Proposition 1. Suppose once the low-fare class is closed it will not reopen, given the game is the dynamic game of complete information between two airlines, Nash equilibrium in stopping regions $(y_A^*(n), y_B^*(n))$ exists **Proof:** From the above analysis, the expected revenue that airline A obtained $F_A(n, y_A)$ is relevant not only with the stopping region itself $y_A^*(n)$ but also with its opponent's stopping policy $y_B^*(n)$, and revenue of airline B depends on both A and B's stopping policies. They want to gain maximum revenue by adjusting their policies.

The optimal policy is to decide the stopping regions of airline A and B. Given the optimal stopping region for airline B is $y_B^*(n)$, we can get the optimal reaction of the stopping region for airline A is $y_A^*(n) = \max\{y_A | G_A(n, y_A) > F_A(n, y_A)\}$. Similarly, given the optimal stopping region for airline A is $y_A^*(n)$, the optimal reaction of the stopping region for airline B is $y_B^*(n) = \max\{y_B | G_B(n, y_B) > F_B(n, y_B)\}$.

If the stopping region's strategies for two airlines are mutually optimal given the other's strategy, in other words, either airline A and B's strategy is mutually optimal response to each other; the strategies are Nash equilibrium. Equilibrium strategy must be the extreme points where its expected revenue reaches the maximum given the opponent's strategy. Nash equilibrium is $(y_A^*(n), y_B^*(n))$.

4. The extension model with vertical diversion

In this section, we analyze the model with buy-up and buy-down, that means vertical diversion of demands. At the earlier periods of the sales horizon, both low-fare and high-fare classes are open to book. Some customers with reservation $v \geq r_l$ (high fare demands) will buy-down with a probability when the low fare is available and the restrictions are acceptable. When low-fare class closed, customers with reservation $r_2 \leq v \leq r_l$ (low fare demands) will buy-up with a probability.

Demand rates from type 1 for low and high-fare classes of flight A are $\lambda_l * [Pr\{r_2 \leq v \leq r_l\} + P_r\{v \geq r_l\} * p_{down}]$, and $\lambda_l * P_r\{v \geq r_l\} * (1 - p_{down})$ when both classes are open. Passengers of type 3 are price sensitive as assumed above; they will buy a low-fare class when it is available, and the demand rate is $\lambda_3 * Pr\{v \geq r_2\} * I_{(P_A < P_B)} + \beta * \lambda_3 * Pr\{v \geq r_2\} * I_{(P_A = P_B)}$.

We can derive the extension model with both horizontal and vertical diversion of demands as follows, let $F'_A(n, y_A)$ be the revenue when low-fare class is still open:

$$F'_A(n, y_A) = P_{AL}^{n'} * [r_2 + F'_A(n-1, y_A-1)] + P_{AH}^{n'} * [r_1 + F'_A(n-1, y_A-1)] + (1 - P_{AL}^{n'} - P_{AH}^{n'}) * F'_A(n-1, y_A) \quad (14)$$

where

$$P_{AL}^{n'} = \lambda_l * [Pr\{r_2 \leq v \leq r_l\} + P_r\{v \geq r_l\} * p_{down}] + \lambda_3 * Pr\{v \geq r_2\} * I_{(P_A < P_B)} + \beta * \lambda_3 * Pr\{v \geq r_2\} * I_{(P_A = P_B)}$$

$$P_{AH}^{n'} = \lambda_l * P_r\{v \geq r_l\} * (1 - p_{down})$$

When the low-fare class is closed, the original demands for low-fare class will buy-up, and the demand rate for high-fare class from type 1 is $\lambda_l * Pr\{r_2 \leq v \leq r_l\} * p_{up}$, from type 3 is $\beta * \lambda_3 * I_{(P_A = P_B)} * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + P_r\{v \geq r_l\}]$. And the expected revenue that airline A can get is as follows:

$$G'_A(n, y_A) = (\lambda_l + \beta * \lambda_3 * I_{(P_A = P_B)}) * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + P_r\{v \geq r_l\}] * [r_1 + G'_A(n-1, y_A-1)] + (1 - (\lambda_l + \beta * \lambda_3 * I_{(P_A = P_B)})) * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + P_r\{v \geq r_l\}] * G'_A(n-1, y_A) \quad (15)$$

Similarly for airline B, $F'_B(n, y_B)$ is the expected revenue when both classes are open:

$$F'_B(n, y_B) = P_{BL}^{n'} * [r_2 + F'_B(n-1, y_B-1)] + P_{BH}^{n'} * [r_1 + F'_B(n-1, y_B-1)] + (1 - P_{BL}^{n'} - P_{BH}^{n'}) * F'_B(n-1, y_B) \quad (16)$$

$$P_{BL}^{n'} = \lambda_2 * [Pr\{r_2 \leq v \leq r_l\} + P_r\{v \geq r_l\} * p_{down}] + \lambda_3 * Pr\{v \geq r_2\} * I_{(P_A > P_B)} + (1 - \beta) * \lambda_3 * Pr\{v \geq r_2\} * I_{(P_A = P_B)}$$

where

$$P_{BH}^{n'} = \lambda_2 * P_r\{v \geq r_l\} * (1 - p_{down})$$

$$G'_B(n, y_B) = (\lambda_2 + (1 - \beta) * \lambda_3 * I_{(P_A = P_B)}) * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + P_r\{v \geq r_l\}] * [r_1 + G'_B(n-1, y_A, y_B-1)] + (1 - (\lambda_2 + (1 - \beta) * \lambda_3 * I_{(P_A = P_B)})) * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + P_r\{v \geq r_l\}] * G'_B(n-1, y_B) \quad (17)$$

Nash equilibrium in stopping low-fare class $(y_A^*(n), y_B^*(n))$ exists, and meets the following

equations:

$$(y_A^{*'}(n), y_B^{*'}(n)) = \begin{cases} y_A^{*'}(n) = \max \{ y_A \mid G_A(n, y_A) > F_A(n, y_A) \} \\ y_B^{*'}(n) = \max \{ y_B \mid G_B(n, y_B) > F_B(n, y_B) \} \end{cases} \quad (18)$$

Proposition 2. Given there are demand diversions of horizontal between airlines and vertical between classes, the optimal stopping points is earlier and the seats left are more than horizontal diversion only, $(y_A^{*'}(n), y_B^{*'}(n)) > (y_A^*(n), y_B^*(n))$.

Proof: Keep both low-fare and high-fare classes open, when there is only horizontal diversion only, a ticket is sold as a low-fare class with a probability P_{AL}^n , and as a high-fare class with probability P_{AH}^n . When incorporated vertical diversion of demand exists between two classes, the probability of selling out a ticket at low-fare class is $P_{AL}^{n'}$, high-fare class $P_{AH}^{n'}$. Obviously, $P_{AL}^{n'} > P_{AL}^n$, $P_{AH}^{n'} < P_{AH}^n$. That means, when there are both horizontal and vertical diversions of demand and there are more requests for the low-fare class and less for the high-fare class, with seats sold at low price more than horizontal diversion only, the expected revenue will decrease. So airlines prefer stopping selling low-fare class early and keeping more seats for high-fare class.

Closing low-fare class and keeping only high-fare class open, purchase rate for high-fare class is $(\lambda_l + \beta * \lambda_3 * I_{(P_A=P_B)}) * \Pr(v \geq r_l)$ when there is horizontal diversion only, and the rate is $(\lambda_l + \beta * \lambda_3 * I_{(P_A=P_B)}) * [Pr\{r_2 \leq v \leq r_l\} * p_{up} + Pr\{v \geq r_l\}]$ with bi-directional diversions. The latter is larger than the former. When low-fare class closed, the original low fare demands buy up to high-fare class, which will enhance the expected revenue. Airlines will stop selling low-fare class earlier and keep more seats for high fare to maximize their revenue.

We can prove the proposition by Littlewood's rule too. Littlewood's rule says keep the discount class open if and only if $r_2 \geq r_l * Pr\{H(n, 0) \geq y\}$ and $Pr\{H(n, 0) \geq y\} \leq \frac{r_2}{r_l}$.

Where y is the number of seats left at period n , and $Pr\{H(n, 0) \geq y\}$ is the probability of more than y passengers arriving between period n and 0. Because the probability of a passenger arriving at each unit period is larger than before, y should be larger.

5. Numerical Examples

For computational purposes, we split the horizon into N periods of equal length. N is large enough so that no more than one demand can occur in one period. Usually, the dilemma that dynamic inventory control models face is computational complexity, especially when N and/or C are/is large. But in our model, the assumption "no-reopen" of low-fare class simplifies computational burden greatly, and the threshold policy decreases the computational complexity greatly. It can be solved with a personal computer efficiently when computing, given the stopping region of one airline and to determine the other airline's optimal strategy.

The algorithms are efficient (see Appendix), and computational results can be calculated in a few minutes with a personal computer when we set $N = 5000, C_A = C_B = 100$. We focus more on the relationships between different variables; to describe the relationships completely, numerical examples are simplified with $N = 100, C_A = C_B = 10$. Price of low-fare and high-fare class is $r_2 = 100$ and $r_1 = 100$, respectively. Demands arrival rate is $\lambda = 22/N$, the ratios of type 1, type 2 and type 3 are $(\alpha_1, \alpha_2, \alpha_3) = (0.2, 0.3, 0.5)$, and $\lambda_1 = \alpha_1 * \lambda$, $\lambda_2 = \alpha_2 * \lambda$, $\lambda_3 = \alpha_3 * \lambda$. Probabilities of reservation prices exceeding the given prices are $F(r_2) = \Pr\{v \geq r_2\} = 0.8$,

$F(r_1) = \Pr\{v \geq r_1\} = 0.3$. Passengers of type 3 choose airline A at probability $\beta = 0.4$ when $P_A = P_B$. The probability that original demands for high-fare class buy down is denoted by $p_{down} = 1 - e^{-\theta * (\frac{r_1}{r_2} - 1)}$, where $\theta = \frac{-\ln 0.5}{frat5_{down} - 1}$, and $frat5_{down}$ is the ratio of the prices when the

probability of buy-down is 50%. Buy-up probability is $p_{up} = e^{-\delta * (\frac{r_1}{r_2} - 1)}$, where $\delta = \frac{-\ln 0.5}{frat5_{up} - 1}$, and $frat5_{up}$ is the ratio of the prices when the probability of buy-up is 50%. For simplicity, let $p_{up} = 1 - p_{down} = 0.5$.

Determine the stopping region

The optimal strategy to stop low-fare class can be calculated by a matlab program based on the model and algorithm derived above. Stopping lines in Figure 4 are an optimal reaction strategy given the other airline's stopping strategy. The upper two lines are stopping lines incorporated into horizontal and vertical diversions of demand, and seats left are more than these considering horizontal diversion only, left on airline B more than airline A given the same stopping point.

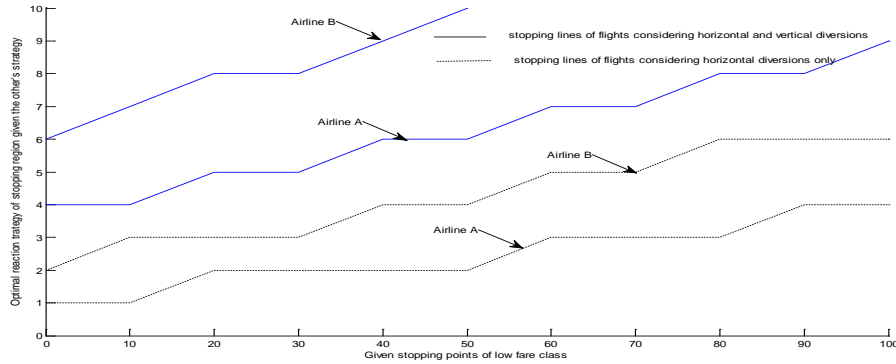


Figure 4. The Optimal Reaction Strategy Given the Rival's Policy

To see relations between stopping region and variables, we kept changing the value of the variables. Table 2 shows the strategy affected by coefficient β , β is changed from 0 to 1, the larger β is, the more passengers will purchase seats on flight A when prices are equal. Stopping region of flight A $y_A^*(n)$ is nondecreasing in β and $y_B^*(n)$ is nonincreasing in β . That means the airline should stop low-fare class earlier and keep more seats for high-fare class if passengers prefer it, excluding price factors

Table 2. Relation of Stopping Regions and β

$\beta =$	Flight A	Flight B
0.0	1	5
0.1	1	5
0.2	1	4
0.3	2	4
0.4	2	4
0.5	2	3
0.6	3	3
0.7	3	3
0.8	3	2
0.9	4	2
1.0	4	2

Table 3. Relation of Stopping Regions and $(\alpha_1, \alpha_2, \alpha_3)$

$(\alpha_1, \alpha_2, \alpha_3)$	Flight A	Flight B
(0.0, 0.5, 0.5)	0	6
(0.1, 0.4, 0.5)	1	5
(0.2, 0.3, 0.5)	2	4
(0.3, 0.2, 0.5)	3	3
(0.4, 0.1, 0.5)	4	2
(0.5, 0.0, 0.5)	5	1
(0.4, 0.6, 0.0)	3	5
(0.3, 0.45, 0.25)	2	4
(0.2, 0.3, 0.5)	2	4
(0.1, 0.15, 0.75)	2	3
(0.0, 0.0, 1.0)	1	3

The ratio of flexible customers and preference coefficient remains unchanged, $\alpha_3 = 0.5$ $\beta = 0.4$ in the upper part of Table 3, the ratios of type 1 and type 2 keep changing. The result shows that the stopping region is increasing in the ratio of their loyal customers. In the lower part of Table 3, let

$\beta = 0.4$ be constant, α_3 is changed from 0 to 1 and the relative proportion of type 1 and type 2 is held; the more flexible the customers the fewer seats kept for the high-fare class.

Stopping the region of Airline A is nondecreasing with the ratio of type 1 (here can be called loyal customers of airline A) α_1 and the ratio β (flexible customers of airline A), but the changing rate caused by α_1 is larger than that caused by β . Airline should increase the proportion of loyal customers by improving competitiveness and attract more flexible customers by pricing policy.

Total expected revenue

In this section, we will discuss the total expected revenue trends with different variables. Take airline A, for example (same to airline B); we considered three states of stopping region given by Airline B: $y_B^*(0) = 0$, $y_B^*(40) = 4$, $y_B^*(100) = 10$. There are three families of curves in figure 3, and solid, dash and dot curves represent the maximum TER with $y_B^*(0) = 0$, $y_B^*(40) = 4$, $y_B^*(100) = 10$, respectively. Each family of curves concludes eleven lines, which means there are 0 to 10 seats left from bottom to up at each period. Conclusions can be drawn from the TER is decreasing with time reduction and increasing with seats left. The value of seats on flights is perishable and decreasing with time. At the beginning of the sales period, more seats can produce more revenue. In another side, the figure shows that, when it comes to a point, the revenues produced by different seats are approximately equal. Airlines are encouraged to open low-fare class at the early sale periods.

(1) Airline A can get more revenue by adjusting its stopping region according to the strategy of Airline B, the earlier Airline B closes low-fare class and the more seats kept for high-fare class, the more TER airline A gains by setting the optimal reaction strategy. We can see dot curves (TER given $y_B^*(100) = 10$, that means there is no low-fare class on flight B) on the top of dash curves (TER given $y_B^*(40) = 4$) and dash curves (TER given $y_B^*(0) = 0$, low-fare class will keep open the whole sales period) on the top of solid curves.

(2) The gaps between every two adjacent lines of the same style represent the expected marginal revenue (MER) $F(n, y) - F(n, y - 1)$. The MER is decreasing when departure time approaches and is decreasing with the number of seats left at period n . So suppliers will sell the seats at a lower price when there are many seats still left with sales period stopping soon.

(3) MER of Airline A increases according to the strategy of Airline B; the earlier stopping period and the more seats kept for high-fare class, the more the MER.

6. Conclusions

The goal of this paper is to analyze the optimal seat allocation policy for the two competing flights, while there are horizontal and vertical diversions of demands caused by competition. We have established a dynamic model for competing airlines engaging in a noncooperative game with complete information. Nash equilibrium of the optimal stopping region exists for both basic the model and the extension model. We have shown that the optimal policy is a threshold policy, which is relatively easy to determine and implement.

Compared with the existing literature, our dynamic model requires the demand of a whole market and the ratios of loyal customers and flexible customers, which can be accessed easily through historical data. We analyzed the demand diversions influencing revenue thoroughly; algorithm and computational speed is high-efficiency.

The contribution of this paper is to consider the dynamic seat allocation problem with more than two flights and two classes completion situation to derive a dynamic model for a multilateral game. Furthermore, it is more interesting to get further research of inventory analysis of the revenue management of train tickets comparing to air tickets. We aim to find the most optimized model of air tickets and train tickets sales to assure that the best sales optimization model can be reached in the perishable air and train tickets' revenue management.

References

- [1] Adida, E., Perakis, G.: Dynamic pricing and inventory control: uncertainty and competition part A: Existence of a Nash equilibrium. Technical report, MIT (2006a)
- [2] Adida, E., Perakis, G.: Dynamic pricing and inventory control: uncertainty and competition part B: An algorithm for the normalized Nash equilibrium. Technical report, MIT (2006b)
- [3] Belobaba, P.P., Wilson, J.L. Impacts of yield management in competitive airline markets. *Journal of Air Transport Management*, 1997,3: 3–9.
- [4] Cooper, W. L. Asymptotic behavior of an allocation policy for revenue management. *Operations Research*, 2002,50(4): 720–727.
- [5] Feng Y, Xiao B. A continuous time yield management model with multiple prices and reversible price changes. *Management Science*, 2000, 46(5) : 644–657.).
- [6] Friesz, T., Mookherjee, R., Rigdon, M.: An evolutionary game theoretic model of revenue management in oligopolistic competition. *J. Revenue Pricing Manag.*4, 156–173 (2005)
- [7] Gallego, G. and M. Hu. Dynamic pricing of perishable assets under competition. *Management Science*, 2014.
- [8] Hu Rong, Xia Hongshan, Jiang Yu. Complex Dynamics for Airlines' Price Competition with Differentiation Strategy. *Journal of Transportation Systems Engineering and Information Technology*, 2013,13(1): 11–16.
- [9] Li, M.Z.F., Oum, T.H., 1998. Seat allocation game on flights with two fares. Working Paper, Nanyang Business School, Nanyang Technological University, Singapore.
- [10] Lilltewood, K. Forecasting and control of passengers. 12th AGIFORS Symposium Proceedings, 1972, 95–117.
- [11] Lippman S. A., McCardle K. F. The competitive news-boy. *Operations Research*, 1997, 45: 54–65.
- [12] Liu, Q. and D. Zhang. Dynamic pricing competition with strategic customers under vertical product differentiation.” *Management Science*, 2013, 59(1): 84–101.
- [13] Mahajan S., Van Ryzin G. J. Inventory competition under dynamic consumer choice. *Operations Research*, 2001, 49:646–657.
- [14] Michael Z. F. Li. Anming Zhang and Yimin Zhang. Airline seat allocation competition. *International Transactions in Operational Research*, 2008,15(4): 439–459.
- [15] Mookherjee, R., Friesz, T.: Pricing, allocation and overbooking in dynamic service network competition when demand is uncertain. Technical report, The Pennsylvania State University, 2005.
- [16] Netessine, S., R. A. Shumsky. Revenue management games: Horizontal and vertical competition. *Management Sci.* 2005,51(5) 813–831.
- [17] P.P. Belobaba. Application of a probabilistic decision model to airline seat inventory control .*Operations Research*, 1989,37: 183–197.
- [18] Parlar M. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistic*, 1998, 35: 397–409.
- [19] Talluri, K. and G. van Ryzin. “Revenue Management Under a General Discrete Choice Model of Consumer Behavior.” *Management Science*, 2004,50(1): 15–33.
- [20] Wen, Z. and Z. Yu-Sheng. A Dynamic Model for Airline Seat Allocation with Passenger Diversion and No-Shows. *Transportation Science*, 2001,35: 80–98.
- [21] Zhao X, Atkins D. Strategic revenue management under price and seat inventory competition. Working Paper, University of British Columbia, 2000